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Riemann normal coordinates: a complete accounting

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Abstract
For the extension of Riemann normal coordinates to higher orders, we show that the amount of geometric information in the kth order for an n-dimensional Riemannian manifold is $F_k^n = \frac{n}{2}\frac{(n+k-1)!}{(n-2)!}\frac{(k-1)}{(k+1)!}$, and we account for this number in terms of the curvature and the Bianchi identities, along with their respective derivatives to various orders.

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1. Introduction

In his classic 1854 presentation *Über die Hypothesen, welche der Geometrie zu Grunde liegen*, Riemann gave us the geometry which bears his name. Effectively (for a translation and detailed discussion of Riemann's presentation, see [1]) he argued that geometry could be described in terms of local coordinates via the line element $\mathrm{d}s^2 = g_{\alpha\beta}(x^\gamma)\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta$. However, one could choose a different set of coordinates: $x^{\bar\mu} = x^{\bar\mu}(x^\alpha)$. Then the $(n/2)(n+1)$ metric tensor coefficients would have the new values

$$g_{\bar\mu\bar\nu} = g_{\alpha\beta}\frac{\partial x^\alpha}{\partial x^{\bar\mu}}\frac{\partial x^\beta}{\partial x^{\bar\nu}}. \tag{1}$$

Riemann essentially considered the Taylor series expansion of the metric around a selected point and argued that one could choose special coordinates (now called *Riemann normal coordinates*) so that (i) to zeroth order the metric had its Euclidean value, (ii) the first-order terms vanished, but (iii) it was generally not possible to make all the second-order terms vanish: they included geometric information parameterized by a fourth rank tensor, now named in his honour the *Riemann curvature tensor*. Briefly,

$$g_{\mu\nu}(0) = \delta_{\mu\nu}, \qquad \partial_\mu g_{\alpha\beta}(0) = 0, \qquad 3\partial^2_{\mu\nu} g_{\alpha\beta}(0) = -(R_{\alpha\mu\beta\nu} + R_{\beta\mu\alpha\nu})(0). \tag{2}$$

Most of the common discussions give this second-order result (see e.g. [2]), but the expansion has been extended to higher orders (in particular, Ni *et al* [3–5] give the expression up to fourth order and explain how it can be systematically iterated; a promising recent work along these lines is [6]). Not surprisingly the higher order terms involve successive covariant derivatives of the curvature tensor.

The question addressed here is simple: how much geometric information is contained in each successive order? Here we find this value and show that, as expected, it exactly corresponds to the amount of independent information in the higher derivatives of the Riemann tensor.

2. Analysis

The kth term in the Taylor series for the metric tensor components expanded about a preselected point (which we take for convenience as the origin) depends on $\partial^k g_{\alpha\beta}(0)$. Since partial derivatives commute we have $S^n_k S^n_2$ components, where $S^n_k := \binom{n+k-1}{k}$ is the number of symmetric ways of choosing k items out of n. (Note that the number of anti-symmetric choices is just $\binom{n}{k}$, the binomial coefficient.) However, we have some coordinate freedom that can be exploited to make a number of these components vanish. Taking the kth derivative of (1), we see that it contains the $k+1$ derivative of the old coordinates with respect to the new. Consequently, we can discount $S^n_{k+1}\binom{n}{1} = n\binom{n+k}{k+1}$ components. Hence, the number of *geometric degrees of freedom* at the kth order is

$$F^n_k := S^n_k S^n_2 - S^n_{k+1} S^n_1 = \frac{n}{2}\frac{(n+k-1)!}{(n-2)!}\frac{(k-1)}{(k+1)!}. \tag{3}$$

This is our first result.

To get some insight into (3), let us note some special cases.

k $= 0$. In this case,

$$F^n_0 = -\binom{n}{2} = -\frac{n}{2}(n-1). \tag{4}$$

This negative value is sensible: there are $S^n_2 = (n/2)(n+1)$ metric conditions and n^2 choices for $\partial x^\alpha/\partial x^{\mu'}$, thus no geometric information at this order but instead *local orthogonal gauge freedom* in the coordinate choice.

k $= 1$. $F^n_1 = 0$ since there are exactly the same number of components of ∂g and $\partial^2 x$, namely $n S^n_2$.

k $= 2$. For this well-known case,

$$F^n_2 = \frac{n^2(n^2-1)}{2\cdot 3!}, \tag{5}$$

which is exactly the number of independent components in the *Riemann curvature tensor* $R_{\alpha\beta\mu\nu}$, as we now verify. Recall that the general curvature tensor has the natural symmetry $R_{\alpha\beta\mu\nu} = R_{\alpha\beta[\mu\nu]}$. However the *Levi-Civita connection* of Riemannian geometry is *metric compatible*, consequently $R_{(\alpha\beta)\mu\nu} \equiv 0$, and *symmetric*, consequently $R^\alpha{}_{[\beta\mu\nu]} \equiv 0$—the *first Bianchi identity*. Hence, the number of independent components of the Riemann tensor is

$$n^2\binom{n}{2} - S^n_2\binom{n}{2} - n\binom{n}{3} \equiv \binom{n}{2}\binom{n}{2} - \binom{n}{3}\binom{n}{1} \equiv F^n_2. \tag{6}$$

k $= 3$. Our formula gives

$$F^n_3 = \frac{n^2(n^2-1)(n+2)}{4!}. \tag{7}$$

Table 1. Some values of F_k^n, the number of kth-order geometric degrees of freedom in n-dimensions.

k	$n = 2$	3	4	5	6	...
0	−1	−3	−6	−10	−15	...
1	0	0	0	0	0	...
2	1	6	20	50	105	...
3	2	15	60	175	420	...
4	3	27	126	420	1134	...
5	4	42	224	840	2520	...
⋮	⋮	⋮	⋮	⋮	⋮	

This is the number of independent components in the covariant derivative of the Riemann tensor $D_\gamma R_{\alpha\beta\mu\nu}$, discounted by the number of independent components of the *second Bianchi identity* $D_{[\gamma} R_{\alpha\beta]\mu\nu}$—after adjustment for those which vanish by virtue of the first Bianchi identity $D_{[\gamma} R_{\alpha\beta\mu]\nu}$:

$$S_1^n F_2^n - \left[\binom{n}{3}\binom{n}{2} - \binom{n}{4}\binom{n}{1}\right] \equiv F_3^n. \tag{8}$$

$\mathbf{k} = 4$. The number of independent components of $D_{(\mu} D_{\nu)} R_{\alpha\beta\gamma\delta}$ must be discounted by the number of terms which vanish identically due to the Bianchi identities: that is the number of $D_\mu D_{[\nu} R_{\alpha\beta]\gamma\delta}$ diminished by the number of $D_\mu D_{[\nu} R_{\alpha\beta\gamma]\delta}$; however, the latter quantity must be corrected by the number of $D_{[\mu} D_\nu R_{\alpha\beta]\gamma\delta}$ diminished by $D_{[\mu} D_\nu R_{\alpha\beta\gamma]\delta}$. In this fashion, we get

$$S_2^n \left[\binom{n}{2}\binom{n}{2} - \binom{n}{3}\binom{n}{1}\right] - S_1^n \left[\binom{n}{3}\binom{n}{2} - \binom{n}{4}\binom{n}{1}\right] + \left[\binom{n}{4}\binom{n}{2} - \binom{n}{5}\binom{n}{1}\right] \equiv F_4^n. \tag{9}$$

From these considerations, one can infer the general pattern for $k \geqslant 2$:

$$\sum_{q=0}^{p} (-1)^q S_{p-q}^n \left[\binom{n}{q+2}\binom{n}{2} - \binom{n}{q+3}\binom{n}{1}\right] \equiv F_{p+2}^n, \tag{10}$$

where the successive terms correspond to

$$D_{(\ldots)}^{p-q} D_{[\ldots}^q (R_{\alpha\beta]\gamma\delta} - R_{\alpha\beta\gamma]\delta}). \tag{11}$$

This accounting has been directly checked for the values of n, k, as indicated in table 1.

3. Final result and conclusion

Remarkably, MAPLE can sum expression (10) for completely general values of n, k to exactly verify the equality of the right-hand side of (10) with (3) to all orders. This equality is our second result. It can be viewed as a demonstration that the geometric information indeed is encoded in the Riemann tensor and its derivatives, just as one would naturally expect on the grounds of basic principles, as well as on the discussions cited above of the explicit form of these expansions.

Since the geometric information available at a point is encoded in the Riemann tensor and its derivatives, scalars constructed from these objects are suitable for use both as geometric coordinates and as candidates for terms in the action of physical theories.

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